Solutions to H.W #5

1. Suppose $s_n \mapsto s$, then for $\epsilon > 0$ there is some $N \in \mathbb{N}$ such that $|s_n - s| < \epsilon$ whenever $n \geq N$. Observe that $\big||s_n| - |s|\big| =$
$= \big||s_n - 0| - |s - 0|\big| \leq |s_n - s| < \epsilon$ (Why?) Thus, $|s_n| \mapsto |s|$.
The converse is, however, not generally true. That is, if $|s_n| \to |s|$, there is no reason to suppose that $s_n \mapsto s$. Take $s_n = (-1)^n$, for instance.

2. $$\lim_{n\to\infty} \sqrt{n^2+n} - n = \lim_{n\to\infty} \frac{n^2+n-n^2}{\sqrt{n^2+n}+n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}+n} =$$
$$= \lim_{n\to\infty} \frac{1}{\sqrt{\frac{n^2+n}{n^2}}+1} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{2}$$

3. Observe that $\cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$. Therefore $2\cos(\frac{\pi}{4}) = s_1$.
Using the identity $\cos(\frac{\theta}{2}) = \sqrt{\frac{1+\cos\theta}{2}}$, notice that
$$\cos\left(\frac{\pi}{8}\right) = \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} = \sqrt{\frac{2+\sqrt{2}}{4}} = \frac{\sqrt{2+\sqrt{2}}}{2} \quad \text{and}$$
$$\cos\left(\frac{\pi}{16}\right) = \sqrt{\frac{1+\cos(\frac{\pi}{8})}{2}} = \sqrt{\frac{2+2\cos(\frac{\pi}{8})}{4}} = \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}.$$
Hence $s_2 = 2\cos(\frac{\pi}{8})$ and $s_3 = 2\cos(\frac{\pi}{16})$. More generally, assuming $s_{n-1} = 2\cos(\frac{\pi}{2^n})$, we get that $s_n = 2\cos(\frac{\pi}{2^{n+1}}) = 2\sqrt{\frac{1+\cos(\frac{\pi}{2^n})}{2}}$
$$= 2\,\frac{\sqrt{2+2\cos(\frac{\pi}{2^n})}}{2} = \sqrt{2+2\cos(\tfrac{\pi}{2^n})} = \sqrt{2+2s_{n-1}}.$$

Clearly, then $s_n = 2\cos(\frac{\pi}{2^{n+1}})$ is an increasing sequence that is bounded above by 2. Notice that $\lim_{n\to\infty} 2\cos(\frac{\pi}{2^{n+1}}) = 2\cos(0) = 2$.
Thus $2 = \sup_{n\in\mathbb{N}} s_n$.

4. First, let's compute a few terms of the sequence.

$s_1 = 0$, $s_2 = 0$, $s_3 = \frac{1}{2}$, $s_4 = \frac{1}{4}$, $s_5 = \frac{1}{2} + \frac{1}{4}$, $s_6 = \frac{1}{4} + \frac{1}{8}$, $s_7 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$,
$s_8 = \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$, $s_9 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$.

More generally,

$$s_{2n+2} = \sum_{k=1}^{n} \left(\frac{1}{2}\right)^{k+1} \text{ and } s_{2n+1} = \sum_{k=1}^{n} \left(\frac{1}{2}\right)^{k} \text{ for } n \geq 1.$$

Hence $\limsup_{n\to\infty} s_n = \lim_{n\to\infty} s_{2n+1} = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k = 1$ and

$$\liminf_{n\to\infty} s_n = \lim_{n\to\infty} s_{2n+2} = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{k+1} = \frac{1}{2}$$

5. Assume first that $\{a_n\}$ and $\{b_n\}$ are bounded above. Then the sets $\{a_k : k \geq n\}$ and $\{b_k : k \geq n\}$ are bounded above for each $n \in \mathbb{N}$. In particular, $\sup_{k\geq n} a_k$ and $\sup_{k\geq n} b_k$ are finite numbers.

Now, $a_n + b_n \leq \sup_{k\geq n} a_k + b_n \leq \sup_{k\geq n} a_k + \sup_{k\geq n} b_k$ implying that $\sup_{k\geq n} a_k + \sup_{k\geq n} b_k$ is an upper bound for the set $\{a_k + b_k : k \geq n\}$ (why?). Thus, $\sup_{k\geq n} (a_k + b_k) \leq \sup_{k\geq n} a_k + \sup_{k\geq n} b_k$ and

$\lim_{n\to\infty} \sup_{k\geq n} (a_k + b_k) \leq \lim_{n\to\infty} \sup_{k\geq n} a_k + \lim_{n\to\infty} \sup_{k\geq n} a_k$ as desired.

If $\limsup_{n\to\infty} a_n = +\infty$ and $\limsup_{n\to\infty} b_n = +\infty$, or $\limsup_{n\to\infty} a_n = -\infty$ and $\limsup_{n\to\infty} b_n = -\infty$, or if only one of the limsup's is unbounded then $\lim_{n\to\infty}\sup(a_n+b_n) = \lim_{n\to\infty}\sup a_n + \lim_{n\to\infty}\sup b_n$.

6. (a) If $a_n = \sqrt{n+1}-\sqrt{n}$ then $A_N = \sum_{n=1}^{N}\left(\sqrt{n+1}-\sqrt{n}\right) =$

$$= \sum_{n=1}^{N}\sqrt{n+1} - \sum_{n=1}^{N}\sqrt{n} = \sum_{n=2}^{N+1}\sqrt{n} - \sum_{n=1}^{N}\sqrt{n} = \sqrt{N+1}-1$$

Hence $\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} A_N = \lim_{N\to\infty}\sqrt{N+1}-1 = \infty$

and the series diverges.

(b) If $a_n = \dfrac{\sqrt{n+1}-\sqrt{n}}{n} = \dfrac{1}{n(\sqrt{n+1}+\sqrt{n})} \leq \dfrac{1}{n(\sqrt{n}+\sqrt{n})} = \dfrac{1}{2n^{3/2}}$

Thus $0 \leq a_n \leq \frac{1}{2n^{3/2}}$ implying that

$\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty}\frac{1}{2n^{3/2}}$, which converges by comparison to a convergent p-series.

(c) If $a_n = (\sqrt[n]{n}-1)^n$, we may apply the root test to see that

$\lim_{n\to\infty}\sup\sqrt[n]{a_n} = \lim_{n\to\infty}\sup(\sqrt[n]{n}-1) = \lim_{n\to\infty}(\sqrt[n]{n}-1) = 0 < 1$.

Hence $\sum_{n=1}^{\infty} a_n$ converges by the root test.

(d) If $a_n = \dfrac{1}{1+z^n}$ for some complex value z, we can break our investigation into 3 cases:

If $|z|>1$, $|1+z^n| \geq |z|^n - 1 > 0$. Since $|z|^n \to \infty$, there is some $N>0$ such that $|z|^n > 2$ whenever $n \geq N$. Hence $\frac{1}{2}|z|^n > 1$ and $-\frac{1}{2}|z|^n < -1$, implying that $|z|^n - \frac{1}{2}|z|^n < |z|^n - 1$ or $\frac{1}{2}|z|^n < |z|^n - 1$. It follows that

$$\frac{1}{|1+z^n|} \leq \frac{1}{|z|^n - 1} \leq \frac{2}{|z|^n}$$

Hence $\displaystyle\sum_{n=N}^{\infty} \frac{1}{|1+z^n|} \leq \sum_{n=N}^{\infty} \frac{2}{|z|^n} < \infty$ and $\sum a_n$ converges absolutely by the comparison test.

If $z=1$, $\displaystyle\lim_{n\to\infty} \frac{1}{1+2^n} = \frac{1}{2} \neq 0$ and the series diverges by the divergence test. If $|z|=1$, $z\neq 1$, $\lim_{n\to\infty} z^n$ does not exist, implying that $\lim_{n\to\infty} a_n \neq 0$

Finally, if $|z|<1$, $\lim_{n\to\infty} z^n = 0$, implying that $\lim_{n\to\infty} a_n = 1$ and the series fails to converge.

Thus $\sum a_n$ converges if and only if $|z|>1$.

7. Assume that $\sum a_n$, $a_n \geq 0$, converges. Then $a = \{a_n\} \in l_1$. We will prove that $\sum \frac{\sqrt{a_n}}{n}$ converges by showing that $b = \left\{\frac{\sqrt{a_n}}{n}\right\}$ is also in l_1. Now,

$$\|\{\sqrt{a_n}\}\|_2 = \sqrt{\sum a_n} < \infty \text{ and } \|\{\tfrac{1}{n}\}\|_2 = \sum \tfrac{1}{n^2} < \infty$$

so, by the Cauchy-Schwarz inequality

$$\sum \frac{\sqrt{a_n}}{n} \leq \sqrt{\sum a_n}\,\sqrt{\sum \tfrac{1}{n^2}} < \infty \text{ and we are done.}$$

8. Let $A_n = \displaystyle\sum_{k=0}^{n} a_k$. Since the series $\sum a_k$ converges, we know that the A_n are bounded. Without loss of generality, the sequence $\{b_n\}_{n=1}^{\infty}$ is decreasing (otherwise, consider $\{-b_n\}$). Define $b = \inf b_n$ and $\beta_n = b_n - b$. Then $\{\beta_n\}_{n=1}^{\infty}$ is also decreasing with $\beta_n \mapsto 0$.

We first show that $\sum a_n \beta_n$ converges by proving that it is Cauchy:

Choose M such that $|A_n| \leq M$ for all n. Given $\epsilon > 0$, there is an integer N such that $\beta_N \leq (\epsilon/2M)$. For $N \leq p \leq q$, we have

$$\left| \sum_{n=p}^{q} a_n \beta_n \right| = \left| \sum_{n=p}^{q-1} A_n(\beta_n - \beta_{n+1}) + A_q \beta_q - A_{p-1}\beta_p \right|$$

$$\leq M \left| \sum_{n=p}^{q-1} (\beta_n - \beta_{n+1}) + \beta_q + \beta_p \right|$$

$$= 2M\beta_p \leq 2M\beta_N \leq \epsilon$$

which proves the desired result. Now, since $\sum a_n$ converges, $\sum b a_n$ converges and $\sum a_n b_n = \sum a_n \beta_n + \sum a_n b$ implying that $\sum a_n b_n$ converges.

9. (a) Applying the root test, we obtain

$$\lim_{n\to\infty} \sup \sqrt[n]{n^3 |z|^n} = \lim_{n\to\infty} \left(\sqrt[n]{n}\right)^3 |z| = |z|.$$

Since $|z| < 1$, implies that the series converges and $|z| > 1$ implies divergence, we see that the radius of convergence is 1,

(b) Applying the ratio test, we obtain

$$\lim_{n\to\infty} \sup \frac{2^{n+1}|z|^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n |z|^n} = \lim_{n\to\infty} \frac{2|z|}{n+1} = 0$$

Hence the radius of convergence is ∞

(c) Applying the root test, we obtain

$$\lim_{n\to\infty} \sup \sqrt[n]{\frac{2^n}{n^2}|z|^n} = \lim_{n\to\infty} \frac{2|z|}{(\sqrt[n]{n})^2} = 2|z|.$$

Since $2|z|<1$ if and only if $|z|<\frac{1}{2}$, we see that the radius of convergence is $1/2$.

(d) Applying the root test, we obtain

$$\lim_{n\to\infty} \sup \sqrt[n]{\frac{n^3}{3^n}|z|^n} = \lim_{n\to\infty} (\sqrt[n]{n})^3 \frac{|z|}{3} = \frac{|z|}{3}.$$

Since $\frac{|z|}{3}<1$ if and only if $|z|<3$, the radius of convergence is 3.

10. Let m be the smallest positive integer such that $|a_k|=m$ for some $k\in\mathbb{N}$. Then, for N large enough, $|a_n|\geq m$ for all $n\geq N$ and therefore $\lim_{n\to\infty}\sup\sqrt[n]{|a_n||z|^n} = \lim_{n\to\infty}\sup\sqrt[n]{|a_n|}\,|z| \geq \lim_{n\to\infty}\sqrt[n]{m}\,|z| =$

$=|z|.$

This means that if $|z|>1$, the series $\sum a_n z^n$ diverges. Hence the radius of convergence is at most 1.

11. (a) Either $\lim_{n\to\infty}\frac{a_n}{1+a_n}=0$ or not. If not, $\sum\frac{a_n}{1+a_n}$ diverges by the divergence test and there is nothing to prove. So assume $\lim_{n\to\infty}\frac{a_n}{1+a_n}=0$. First notice that this implies that $\lim_{n\to\infty}a_n=0$:

Define $f:[0,1)\to\mathbb{R}$ by $f(x)=\frac{x}{1-x}$. Then f is continuous and $f\left(\frac{a_n}{1+a_n}\right)=a_n$. Now $\lim_{n\to\infty}a_n = \lim_{n\to\infty} f\left(\frac{a_n}{1+a_n}\right) = f\left(\lim_{n\to\infty}\frac{a_n}{1+a_n}\right) =$

$= f(0) = 0$

Since $a_n > 0$, $\frac{a_n}{1+a_n} < a_n$ and $\lim_{n\to\infty} \frac{\frac{a_n}{1+a_n}}{a_n} =$

$= \lim_{n\to\infty} \frac{1}{1+a_n} = 1.$

Hence, for $0 < \epsilon < 1$, there exists an $N \in \mathbb{N}$ such that

$$\frac{\frac{a_n}{1+a_n}}{a_n} > 1-\epsilon > 0$$

for all $n \geq N$. In particular, $\frac{a_n}{1+a_n} > (1-\epsilon)a_n$ for all $n \geq N$. Thus, $\sum_{n=N}^{\infty} \frac{a_n}{1+a_n} \geq \sum_{n=N}^{\infty} (1-\epsilon)a_n \mapsto \infty$ by hypothesis

(series of positive terms can only diverge to infinity).

(b) Since the $a_n > 0$ and $\sum a_n$ diverges, it is clear that the S_n form an increasing sequence with $\lim_{n\to\infty} S_n = \infty$. We will show that $\sum \frac{a_n}{S_n}$ diverges by proving that it is not Cauchy. Equivalently, we will show that the tail of the series, $\sum_{k=N}^{\infty} \frac{a_k}{S_k}$, does not go to 0 as $N \mapsto \infty$.

Now $\frac{a_{N+1}}{S_{N+1}} + \dots + \frac{a_{N+k}}{S_{N+k}} \geq \frac{a_{N+1}}{S_{N+k}} + \dots + \frac{a_{N+k}}{S_{N+k}} =$

$$= \frac{a_{N+1} + \dots + a_{N+k}}{S_{N+k}} = \frac{S_{N+k} - S_N}{S_{N+k}} = 1 - \frac{S_N}{S_{N+k}}$$

Hence, $\lim_{k\to\infty} \left(\sum_{n=N}^{K} \frac{a_n}{S_n} \right) = \lim_{k\to\infty} \left(\sum_{n=1}^{K} \frac{a_{N+n}}{S_{N+n}} \right) \geq \lim_{k\to\infty} 1 - \frac{S_N}{S_{N+k}} = 1$

which proves that $\sum_{n=N}^{\infty} \frac{a_n}{s_n} \geq 1$ for all N. Thus

$\lim_{N\to\infty} \sum_{n=N}^{\infty} \frac{a_n}{s_n} \geq 1$ as desired.

(c) Since the s_n are increasing, $s_n^2 \geq s_n s_{n-1}$ so

$$\frac{a_n}{s_n^2} \leq \frac{a_n}{s_n s_{n-1}} = \frac{s_n - s_{n-1}}{s_n s_{n-1}} = \frac{1}{s_{n-1}} - \frac{1}{s_n}.$$

Now,

$$\sum_{n=2}^{N} \frac{a_n}{s_n^2} \leq \sum_{n=2}^{N} \left(\frac{1}{s_{n-1}} - \frac{1}{s_n} \right) = \sum_{n=1}^{N-1} \frac{1}{s_n} - \sum_{n=2}^{N} \frac{1}{s_n} =$$

$$= \frac{1}{s_1} - \frac{1}{s_N} \longmapsto \frac{1}{s_1} \text{ as } N \longmapsto \infty \text{ because } \lim_{N\to\infty} s_N = \infty.$$

Hence $\sum_{n=2}^{\infty} \frac{a_n}{s_n^2} \leq \frac{1}{s_1} = \frac{1}{a_1}$ which shows that the series converges.

(d) The series $\sum \frac{a_n}{1+na_n}$ diverges: Either $a_n > \frac{1}{n\ln n}$ for all n bigger than some N or not. If $a_n > \frac{1}{n\ln n}$ then

$$\frac{a_n}{1+na_n} > \frac{\frac{1}{n\ln n}}{1+\frac{1}{\ln n}}$$

(because $f(x) = \frac{x}{1+nx}$ is an increasing function). Now

$$\frac{\frac{1}{n\ln n}}{1+\frac{1}{\ln n}} = \frac{1}{n\ln n + n} \geq \frac{1}{n\ln n + n\ln n} = \frac{1}{2n\ln n}$$

Hence $\sum_{n=N}^{\infty} \frac{a_n}{1+na_n} \geq \sum_{n=N}^{\infty} \frac{1}{2n\ln n} = \infty$

If $a_n \leq \frac{1}{n\ln n}$ then $\lim_{n\to\infty} na_n \leq \lim_{n\to\infty} \frac{1}{\ln n} = 0$

Hence $\lim_{n\to\infty} \dfrac{\frac{a_n}{1+na_n}}{a_n} = \lim_{n\to\infty} \dfrac{1}{1+na_n} = 1$.

In particular, for $0<\epsilon<1$, $n \geqslant N$ implies that $\frac{1}{1+na_n} > 1-\epsilon$

or $\frac{a_n}{1+na_n} > (1-\epsilon)a_n$

Thus $\sum_{n=N}^{\infty} \frac{a_n}{1+na_n} > \sum_{n=N}^{\infty} (1-\epsilon)a_n$ and the series $\sum \frac{a_n}{1+na_n}$

diverges by comparison with the divergent series $\sum (1-\epsilon)a_n$.

On the other hand, the series $\sum \frac{a_n}{1+n^2a_n} = \sum \frac{n^2a_n}{1+n^2a_n}\frac{1}{n^2}$

$\leqslant \sum \frac{1}{n^2}$ converges by comparison to $\sum \frac{1}{n^2}$.

12. (a) Let $r_n = \sum_{k=n} a_k$. Since $\sum a_k$ converges,

$\lim_{n\to\infty} r_n = 0$. Now $\sum_{k=m}^{n} \frac{a_k}{r_k} = \frac{a_m}{r_m} + \dots + \frac{a_n}{r_n} >$

$> \frac{a_m + \dots + a_n}{r_m}$, because $a_k > 0$ implies that $r_n < r_m$ if

$n > m$.

But $\frac{a_m+\dots+a_n}{r_m} = \frac{\sum_{k=m}^{n} a_k}{r_m} = \frac{\sum_{k=m}^{\infty} a_k - \sum_{k=n+1}^{\infty} a_k}{r_m} =$

$= \frac{r_m - r_{n+1}}{r_m} > \frac{r_m - r_n}{r_m} = 1 - \frac{r_n}{r_m}$

Thus $\sum_{k=m}^{\infty} \frac{a_k}{r_k} \geqslant \lim_{n\to\infty} 1 - \frac{r_n}{r_m} = 1$, for all m.

It follows that the sequence $s_N = \sum_{k=1}^{N} \frac{a_k}{r_k}$ is not

Cauchy, from which it follows that s_N fails to converge.

(b) $$\frac{a_n}{\sqrt{r_n}} = \frac{r_n - r_{n+1}}{\sqrt{r_n}} = \frac{(\sqrt{r_n} - \sqrt{r_{n+1}})}{\sqrt{r_n}}(\sqrt{r_n} + \sqrt{r_{n+1}}) =$$

$$= \frac{\sqrt{r_n} + \sqrt{r_{n+1}}}{\sqrt{r_n}}(\sqrt{r_n} - \sqrt{r_{n+1}}) = \left(1 + \frac{\sqrt{r_{n+1}}}{\sqrt{r_n}}\right)(\sqrt{r_n} - \sqrt{r_{n+1}}) <$$

$$< 2(\sqrt{r_n} - \sqrt{r_{n+1}}) \quad \text{Since } r_{n+1} < r_n.$$

Hence,

$$\sum_{n=1}^{N} \frac{a_n}{\sqrt{r_n}} < \sum_{n=1}^{N} 2(\sqrt{r_n} - \sqrt{r_{n+1}}) = 2(\sqrt{r_1} - \sqrt{r_{N+1}})$$

So

$$\sum_{n=1}^{\infty} \frac{a_n}{\sqrt{r_n}} = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{a_n}{\sqrt{r_n}} \leq \lim_{N\to\infty} \sum_{n=1}^{N} 2(\sqrt{r_n} - \sqrt{r_{n+1}}) =$$

$$= \lim_{N\to\infty} 2(\sqrt{r_1} - \sqrt{r_{N+1}}) = 2\sqrt{r_1}$$

13. (a) Assume $\lim_{n\to\infty} s_n = s$. Then for $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that $|s_n - s| < \epsilon$ whenever $n \geq N$

Now $$|\sigma_n - s| = \left| \frac{\sum_{k=0}^{n} s_k - (n+1)s}{n+1} \right| = \frac{\left|\sum_{k=0}^{n}(s_k - s)\right|}{n+1}$$

$$\leq \frac{\left|\sum_{k=0}^{N}(s_k - s)\right|}{n+1} + \frac{\sum_{k=N+1}^{n}|s_k - s|}{n+1} \leq \frac{\left|\sum_{k=0}^{N}(s_k - s)\right|}{n+1} + \frac{(n-N)\epsilon}{n+1}$$

Hence $$\lim_{n\to\infty}|\sigma_n - s| \leq \lim_{n\to\infty} \frac{\left|\sum_{k=0}^{N}(s_k - s)\right|}{n+1} + \lim_{n\to\infty} \frac{(n-N)\epsilon}{n+1} = \epsilon$$

Since ϵ is arbitrary, it follows that $\lim_{n\to\infty} |\sigma_n - s| = 0$

By the results of exercise 1, we may conclude that $\lim_{n\to\infty} \sigma_n - s = 0$

(b) Let $s_n = (-1)^n$. Then

$$\left| \sum_{k=0}^{n} s_n \right| \leq 1 \text{ for all } n.$$

Therefore $\lim_{n\to\infty} \sigma_n = 0$ even though s_n fails to converge.

(c) Yes it can happen!

Set $s_0 = 1$, $s_1 = \frac{1}{2}$, $s_2 = 2$, $s_3 = \frac{1}{2^2}$, $s_4 = 3$, $s_5 = \frac{1}{2^3}$, $s_6 = \frac{1}{2^4}$, $s_7 = \frac{1}{2^5}$, $s_8 = 4$.

More generally,

$$s_n = \begin{cases} p+1 & \text{if } n = 2^p \\ \left(\frac{1}{2}\right)^{2^p + k - \frac{p(p+1)}{2}} & \text{if } n = 2^p + k,\ 1 \leq k < 2^p \end{cases}$$

Let $2^p \leq n < 2^{p+1}$. Then

$$\sigma_n = \frac{\sum_{j=0}^{n} s_j}{n+1} \leq \frac{\sum_{j=1}^{\infty} \left(\frac{1}{2}\right)^j + \sum_{k=0}^{p} (k+1)}{2^p + 1} = \frac{1 + \frac{(p+1)(p+2)}{2}}{2^p + 1}$$

Clearly then $\lim_{n\to\infty} \sigma_n \leq \lim_{p\to\infty} \frac{1 + \frac{(p+1)(p+2)}{2}}{2^p + 1} = 0$

Since each $s_n > 0$, $\sigma_n \geq 0$ so $\lim_{n\to\infty} \sigma_n = 0$. Notice however that $\lim_{n\to\infty} \sup s_n = \lim_{p\to\infty} p = \infty$

(d) Set $a_n = s_n - s_{n-1}$, $n \geq 1$. Then $s_n - \sigma_n =$

$$= s_n - \frac{1}{n+1}\sum_{k=0}^{n} s_k = \frac{1}{n+1}\left((n+1)s_n - \sum_{k=0}^{n} s_k\right) = \frac{1}{n+1}\left(\sum_{k=0}^{n} s_n - \sum_{k=0}^{n} s_k\right)$$

$$= \frac{1}{n+1}\sum_{k=0}^{n}(s_n - s_k)$$

Observe that for $k < n$, $s_n - s_k = (s_n - s_{n-1}) + (s_{n-1} - s_{n-2}) + \ldots +$
$+ (s_{k+1} - s_k) = a_n + \ldots + a_k$.

Hence $s_n - \sigma_n = \frac{1}{n+1}\sum_{k=1}^{n}(a_n + \ldots + a_k) =$

$$= \frac{1}{n+1}\left([a_n + \ldots + a_1] + [a_n + \ldots + a_2] + \ldots + [a_n + a_{n-1}] + [a_n]\right)$$

$$= \frac{1}{n+1}\left(n a_n + (n-1)a_{n-1} + \ldots + a_1\right) = \frac{1}{n+1}\sum_{k=1}^{n} k a_k$$

If $\lim_{n\to\infty} n a_n = 0$, for each $\epsilon > 0$ there is some N such that $n > N$ implies that $|n a_n| < \epsilon$. Then $\lim_{n\to\infty} \frac{1}{n+1}\left|\sum_{k=1}^{n} k a_k\right| \leq$

$$\leq \lim_{n\to\infty} \frac{1}{n+1}\sum_{k=1}^{n}|k a_k| \leq \lim_{n\to\infty}\frac{1}{n+1}\left(\sum_{k=1}^{N}|k a_k| + \sum_{k=N+1}^{n}\epsilon\right) =$$

$$= \lim_{n\to\infty}\frac{1}{n+1}\left(\sum_{k=1}^{N}|k a_k| + (n-N)\epsilon\right) = \epsilon$$

Since ϵ is arbitrary, it follows that $\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=1}^{n} k a_k = 0$

If, furthermore, $\lim_{n\to\infty}\sigma_n$ exists and equals σ then

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}(s_n - \sigma_n) + \lim_{n\to\infty}\sigma_n = \lim_{n\to\infty}\frac{1}{n+1}\sum_{k=1}^{n} k a_k + \sigma = \sigma.$$

(e) If $m<n$, then

$$s_n-\sigma_n = \frac{1}{n+1}\left((n+1)s_n - \sum_{k=0}^{n} s_k\right) = \frac{1}{n+1}\left((n+1)s_n - \sum_{k=0}^{m} s_k - \sum_{k=m+1}^{n} s_k\right)$$

$$= \frac{1}{n+1}\left(\left[(m+1)s_n - \sum_{k=0}^{m} s_k\right] + \left[(n-m)s_n - \sum_{k=m+1}^{n} s_k\right]\right)$$

$$= \frac{1}{n+1}\left((m+1)(s_n-\sigma_m)) + \sum_{k=m+1}^{n}(s_n-s_k)\right)$$

Thus $(n+1)(s_n-\sigma_n) = (m+1)(s_n-\sigma_m) + \sum_{k=m+1}^{n}(s_n-s_k)$

and hence $(n+1)(s_n-\sigma_n) = (m+1)(s_n-\sigma_n) + (n-m)(s_n-\sigma_n) =$

$= (m+1)(s_n-\sigma_m) + \sum_{k=m+1}^{n}(s_n-s_k)$

or $(n-m)(s_n-\sigma_n) = (m+1)(s_n-\sigma_m) - (m+1)(s_n-\sigma_n) + \sum_{k=m+1}^{n}(s_n-s_k)$

$= (m+1)(s_n-\sigma_m) + (m+1)(\sigma_n-s_n) + \sum_{k=m+1}^{n}(s_n-s_k) =$

$= (m+1)(\sigma_n-\sigma_m) + \sum_{k=m+1}^{n}(s_n-s_k)$

and it follows that

$$s_n-\sigma_n = \frac{m+1}{n-m}(\sigma_n-\sigma_m) + \frac{1}{n-m}\sum_{k=m+1}^{n}(s_n-s_k)$$

Now assume that $a_n = s_n - s_{n-1}$ and $|n a_n| \leq M$. Notice that

$|s_n-s_k| \leq |s_n-s_{n-1}| + |s_{n-1}-s_{n-2}| + \ldots + |s_{k+1}-s_k| =$

$$= \frac{|n a_n|}{n} + \frac{|(n-1)a_{n-1}|}{n-1} + \ldots + \frac{|(k+1)a_{k+1}|}{k+1} \leq \frac{|n a_n|}{k+1} + \ldots + \frac{|(k+1)a_{k+1}|}{k+1}$$

$$\leq \frac{M}{k+1} + \ldots + \frac{M}{k+1} = \frac{(n-k)M}{k+1}$$

Since the chain of sums is longest when $k=m+1$, it follows that $\frac{(n-k)M}{k+1} \leq \frac{(n-(m+1))M}{(m+1)+1} = \frac{(n-m-1)M}{m+2}$

Hence

$$\frac{1}{n-m}\sum_{k=m+1}^{n} |s_n - s_k| \leq \frac{1}{n-m}\sum_{k=m+1}^{n} \frac{(n-m-1)M}{m+2} = \frac{n-m-1}{m+2} M$$

Let $\epsilon > 0$. For each n, fix m such that

$\frac{n-m-1}{m+2} = \frac{n-(m+1)}{(m+1)+1} < \epsilon$. Then $\frac{n-\epsilon}{1+\epsilon} < m+1$ and since we require $m<n$, we can impose the restriction $m \leq \frac{n-\epsilon}{1+\epsilon} < n$. Also, since $f(x) = \frac{n-x}{1+x}$ is a decreasing function, it follows that $\frac{n-m}{m+1} < \frac{n-m-1}{m+2} < \epsilon$.

Hence $\frac{m+1}{n-m} > \frac{1}{\epsilon}$.

Thus

$$|s_n - \sigma_n| \leq \frac{m+1}{n-m}|\sigma_n - \sigma_m| + \frac{1}{n-m}\sum_{k=m+1}^{n}|s_n - s_k| \leq$$

$$\leq \frac{m+1}{n-m}|\sigma_n - \sigma_m| + \frac{(n-m-1)}{m+2}M < \frac{1}{\epsilon}|s_n - \sigma_m| + \epsilon M$$

Notice however that $m \leq \frac{n-\epsilon}{1+\epsilon} < m+1$ implies that $m = m(n)$ is an increasing function of n. Thus the fact that σ_n is Cauchy allows us to pick n large enough such that m is also large and $|\sigma_n - \sigma_m| < \epsilon^2$. For these values we get

$|s_n - \sigma_n| \leq \frac{1}{\epsilon}\epsilon^2 + \epsilon M = \epsilon + \epsilon M = \epsilon(1+M)$. Since ϵ is arbitrary we see that $\limsup_{n\to\infty} |s_n - \sigma_n| = 0$ from which the desired result is obtained.